Large-scale statistics of the Kuramoto-Sivashinsky equation: A wavelet-based approach

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We show that the statistical properties of the large scales of the Kuramoto-Sivashinsky equation in the extended system limit can be understood in terms of the dynamical behavior of the same equation in a small finite domain. Our method relies on the description of the solutions of this equation in terms of wavelets, and allows us to model the energy transfer between small and large scales. We show that the effective equation obtained in this way can be consistently approximated by a forced Burgers equation only for scales far from the cutoff between small and large wavelengths. [S1063-651X(96)01407-9]

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I. INTRODUCTION

In recent years considerable effort has been devoted to the study of effective models for deterministic dynamical systems with many degrees of freedom. One of the goals of these efforts is to describe statistical properties of the system through those of a coarse-grained model corresponding to the large scale dynamics. In such an analysis, one realizes that the net effect of the small scales is twofold: (i) the renormalization of the coupling terms and (ii) the appearance of an additive stochastic noise. The key issue in any approach to this problem is how to deal properly with the statistics of the small scales as well as their coupling to the large scales. In this paper, we will focus on the Kuramoto-Sivashinsky (KS) [1] equation

$$\partial_t u + L^{-2} u_{xx} + L^{-4} u_{xxxx} + L^{-1} u u_x = 0 \tag{1}$$

on the one-dimensional (1D) periodic domain $[0,1]$. This equation is particularly interesting in view of the recent attention [2,3] devoted to the connection between its long wavelength properties and the KPZ equation [4] for interface growth. Here, L is the single control parameter of the system, and we will be concerned with the régime $L \to \infty$, which corresponds to the extended system limit. Several papers have addressed the study of the long wavelength properties of this equation. Yakhot [5] used a perturbative renormalization group (RNG) [5,6] approach, and suggested that the KS equation can be described at long wavelengths by the stochastic Burgers equation

$$\partial_t u - \nu L^{-2} u_{xx} + L^{-1} u u_x = \eta(x,t), \quad \nu > 0,$$

where $\eta(x,t)$ is the stochastic forcing. Although this method is constructive in the sense that it provides an effective expression for the renormalized viscosity ν, the introduction of unknown parameters in the description of the small scales makes the effective computation of ν unclear.

More recently, Procaccia and co-workers [7] studied the existence of scale-invariant solutions of the KS and KPZ equations in d dimensions using a renormalized perturbation theory based on the Dyson-Wild equations for the Green's function and the correlation function. The main result of this theory is the computation of the exponent z such that the steady-state correlations of the discrete Fourier components $\hat{h}(q,t)$, $q = 2\pi n/L$, $|n| = 1, \ldots, L/2$ of the height function $h(x,t) = \int_0^x u(s,t)\,ds$ read

$$\langle \hat{h}(q,t)\hat{h}(-q,s)\rangle = \frac{A}{Lq^2} g(q^z|t-s|), \tag{2}$$

where A is a constant, and $g(x)$ is a universal function such that $g(0) = 1$ and $\lim_{x\to\infty} g(x) = 0$. As shown in Ref. [2], one has to integrate Eq. (1) for rather large values of $L(\sim 4000)$ in order to verify the scaling relation (2).

A completely different way of thinking has been initiated by Zaleski [8] and later pursued by Hayot, Jayaprakash, and co-workers [3] in one and two dimensions. These authors use a coarse-graining procedure to integrate out short wavelength degrees of freedom $k > \Lambda$, for a suitable cutoff Λ. If $u(k,t)$ denotes the Fourier transform of $u(x,t)$, then Eq. (1) may be rewritten as

$$u_t(k,t) = -\nu 4\pi^2 \frac{k^2}{L^2} u(k,t) + g(k,t) + f(k,t), \tag{3}$$

where

$$g(k,t) = -i\frac{2\pi k}{L} \sum_{|q|<\Lambda, |k-q|<\lambda} u(q,t)u(k-q,t) \tag{4}$$

is the Fourier transform of the nonlinear term, involving only large-scale components. The value of the renormalized viscosity ν is determined in such a way that the ''stochastic'' forcing

$$f(k,t) = \left(\nu + \frac{1}{L^2} - \frac{4\pi^2}{L^2}\right)\frac{4\pi^2 k^2}{L^2} u(k,t) - i\frac{2\pi k}{L} \sum_{|q|\geq\Lambda \text{ or } |k-q|\geq\Lambda} u(q,t)u(k-q,t) \tag{5}$$

involving the coupling between at least one small-scale mode and any other mode, is uncorrelated with the large-scale modes for time differences longer than the (shortest) characteristic time scale of the large-scales τ_l

$$T^{-1}\int_0^T f(k,t)u(-k,t-\tau)dt \to 0 \text{ as } T\to\infty,$$

$$\tau > \tau_l, \quad |k| < \Lambda. \tag{6}$$

In other words, the assumption that only the linear term is renormalized yields a closed expression for ν in terms of three point correlation functions mixing short and long wavelength modes. It then remains to check that the above definition of ν is independent of the parameters Λ, k, and the time delay τ, and that the ''stochastic forcing'' $f(k,t)$ has the expected correlation properties. Although this method works well, it is not constructive, in the sense that one effectively needs to integrate the full KS equation in order to compute these three point correlations, and eventually ν.

The main motivation of the present paper was to model the energy transfer between large and small scales in the KS equation, in order to gain some insight into the mechanisms responsible for the renormalization of the viscosity. More precisely, we show that the two and three point correlation functions mixing both large and small scales can be numerically computed from the response of a low dimensional dynamical system, namely, the KS equation for small values of L, to an external forcing. Our procedure does not yield a closed analytic form for the coarse-grained equations, but it provides a completely explicit recipe free of unknown parameters, and easily amenable to numerical computation. In the process of (re)writing this paper, we learned of a similar paper by C. Chow and T. Hwa [9], leading to the same result as ours, but using completely different methods, and, in our opinion, requiring stronger assumptions about the behavior of small scales. We will comment on the approach of [9] throughout the following. We also refer the reader to Refs. [10–12] for other approaches similar to ours.

Another interesting question, which to the best of our knowledge has not been addressed in the context of the KS equation, concerns the behavior of the coarse-grained equation for the scales close to the arbitrary cutoff separating small from large scales. One expects that approximations relying on differences in time scales will break down in the vicinity of this boundary. This point will be considered in the context of our particular modeling of the small-scale behavior.

The rest of the paper is organized as follows. In Sec. II, we describe the main analytic tool of our approach, namely, the use of an orthogonal wavelet basis, and we study the structure of the equations generated by projecting the KS equation onto such a basis. In Sec. III, we explain our model for the coupling between small and large scales and we describe and compare the results of three numerical simulations: (1) of the full KS equation on a ''long'' domain $L=400$; (2) of eight independent ''short'' systems ($L=400/8=50$) forced by the large scales of the full simulation; and (3), of a closed system of coupled large and independent small-scale systems. Section IV contains concluding remarks.

II. WAVELET PROJECTIONS OF THE KS EQUATION

Our approach is based on the use of a particular orthogonal *wavelet* basis [13]. The construction of this basis relies on the existence of a function $\Psi(x)$, which completely defines its elements, such that the infinite set $\{\Psi_{j,k}, j=0,1,\ldots,k=0,1,\ldots,2^j-1\}$ forms a basis for the zero mean, finite energy periodic functions on $[0,1]$. The relation between $\Psi(x)$ and $\Psi_{j,k}(x)$ is given by

$$\Psi_{j,k}(x)=2^{j/2}\sum_{n=-\infty}^{\infty}\Psi(2^j(x+n)-k). \tag{7}$$

Typically, $\Psi(x)$ is a rapidly (exponentially) decreasing function [14], which can effectively be considered as compactly supported for numerical purposes. In this paper, we use a fifth degree spline [15] for $\Psi(x)$, although this fact will not directly enter our reasoning below. Then one can easily see from relation (7) that

$$\Psi_{j,k}(x)=2^{j/2}\Psi(2^jx-k)+O(\exp(-2^j/S)) \tag{8}$$

provided $\Psi(x)\sim\exp(-|x|/S)$ as $x\to\infty$. In physical terms, the wavelet basis is built of rescaled and translated versions of the $\Psi(x)$ function so that, for each scale j, we obtain a set of 2^j functions, centered at the points $(k+0.5)2^{-j}$, $k=0,1,\ldots,2^j-1$ [13,15], and sufficiently rapidly decaying that their supports are concentrated in intervals of length $\sim 2^{-j}$. In the Fourier domain, a wavelet basis corresponds to a logarithmic partition so that the support of any of the 2^j Fourier transforms $\hat{\Psi}_{j,k}(q)$ is centered around the wave vector $q_0 2^j$, where $q_0\sim 1$ corresponds to the maximum of the modulus of the Fourier transform $\hat{\Psi}(x)$. The decay in the wave number is algebraic [15].

As usual, we represent the solution of the KS equation in this basis via

$$u(x,t)=\sum_{\alpha} a_\alpha(t)\Psi_\alpha(x), \tag{9}$$

where α denotes the multi-index (j,k). We will also introduce an arbitrary scale j_0 such that $\{a_\alpha^< = a_{j,k}, j\leqslant j_0\}$ represents the large-scale component of the solution, and $\{a_\alpha^> = a_{j,k}, j>j_0\}$ represents the small-scale component (the role of j_0 is completely analogous to that of the cutoff wave number Λ defined via $\Lambda\sim 2^{j_0}q_0$). Specific values of j_0 will be fixed later.

Several observations suggest that, for the present application, wavelets are superior to standard orthogonal bases (such as Fourier modes). In particular: (i) due to the exponential decrease in the scales between wavelet ''generations'' inherent in the wavelet basis *for equations such as the KS equation in the regions of the spectrum we are interested in*, the characteristic time scale of all the $a_{j,k}$'s is the same for each fixed j, but differs by an order of magnitude from that of the neighboring $a_{j+1,k}$ scales. An analogous partition of Fourier modes $e^{2i\pi kx}$ into shells of the form $k_0^n\leqslant|k|<k_0^{n+1}$ has been systematically used in RNG calculations [5,6,16], but while in the Fourier case there is no clear gap between time correlations of modes in neighboring shells, for wavelets, all the coefficients at the same scale have the same time correlation

and neighboring scales differ by a factor of 2. This point is crucial when invoking any kind of multiple time scale technique, as is the case in the RNG method. (ii) Due to the particular structure of the wavelet basis, it will be shown below that the equations governing the time evolution of the wavelet coefficients a_α for values of $j>j_0$ can be decomposed into a set of independent replicas of the equations computed from the wavelet projection of the KS equation for a smaller value of the parameter L. All the results in this paper rely strongly on this observation, which we now explain.

Let us write Eq. (1) in terms of the a_α variables

$$\dot{a}_\alpha = l_{\alpha\beta}a_\beta + n_{\alpha\beta\gamma}a_\beta a_\gamma, \tag{10}$$

where $l_{\alpha\beta}=\int_0^1 \Psi_\alpha(x)(-L^{-2}\partial_{xx}-L^{-4}\partial_{xxxx})\Psi_\beta(x)dx$, etc., $\ldots$, and summation over repeated indices is implied. We introduce the splitting between large ($a^<$) and small ($a^>$) scales, and rewrite the evolution equation of the small scales as

$$\dot{a}^>_\alpha = [l^{>>}_{\alpha\beta}+\delta l^{>>}_{\alpha\beta}(t)]a^>_\beta + n^{>>>}_{\alpha\beta\gamma}a^>_\beta a^>_\gamma + f^>_\alpha(t), \tag{11}$$

where $\delta l^{>>}_{\alpha\beta}(t)=n^{>><}_{\alpha\beta\gamma}a^<_\gamma + n^{><>}_{\alpha\gamma\beta}a^<_\gamma$ is a slowly varying perturbation of the linear term, and $f^>_\alpha(t) = n^{><<}_{\alpha\beta\gamma}a^<_\beta a^<_\gamma + l^{><}_{\alpha\beta}a^<_\beta$ is a slowly varying additive forcing term.

In this section, we consider the small-scales equation (11) without the forcing terms $\delta l^{>>}_{\alpha\beta}$ and $f^>_\alpha$, i.e., in isolation from the large scales. The key observation is that the equations governing the statistical properties of the $a^>_\alpha$ variables satisfying these unforced equations (11) are identical to those of the KS equation with parameter $\tilde{L}=L2^{-(j_0+1)}$, up to rescaling and small corrections.

Namely, if we define the ''boxes'' B_k, $k=0,\ldots,2^{j_0+1}-1$, specified by the sets of indices (j,k') with $j\geqslant j_0+1$ and $|k2^{-(j_0+1)}-k'2^{-j}|\leqslant 2^{-(j_0+1)}$, we can decouple the $a^>_\alpha$ equations into 2^{j_0+1} independent sets of equations, each one including only variables $a^>$ belonging to the same B_k box, in a manner that preserves the statistics but not necessarily the dynamics. The reason for this is easily seen from inspection of the moment equations associated with the unforced equation (11) ($f^>_\alpha=\delta^{>>}_{\alpha\beta}=0$). For example, consider the equations of motion for the second moment

$$\frac{d}{d\tau}\langle a^>_\alpha(t)a^>_\beta(t+\tau)\rangle = \langle a^>_\alpha(t)l^{>>}_{\beta\gamma}a^>_\gamma(t+\tau)\rangle + \langle a^>_\alpha(t)n^{>>>}_{\beta\gamma\delta}a^>_\gamma(t+\tau)a^>_\delta(t+\tau)\rangle, \tag{12}$$

where $\langle\cdot\rangle$ is the time average. Let us denote by $|\alpha-\beta|$ the distance $|k2^{-j}-k'2^{-j'}|$ (modulo 1), where $\alpha=(j,k)$ and $\beta=(j',k')$. Using the estimate $\Psi(x)\sim\exp(-|x|/S)$ as $x\to\infty$, it is straightforward to obtain the following inequalities:

$$|l_{\alpha\beta}|\leqslant C_1 2^{|j-j'|/2}(\exp^{-|\alpha-\beta|/S}+\exp^{-(1-|\alpha-\beta|)/S})$$

and

$$|n_{\alpha\beta\gamma}|\leqslant C_2 2^{|j-j'|/2}(\exp^{-|\alpha-\beta|/S} + \exp^{-(1-|\alpha-\beta|)/S})2^{|j-j''|/2}(\exp^{-|\alpha-\gamma|/S} + \exp^{-(1-|\alpha-\gamma|)/S}),$$

where C_1 and C_2 are two constants. We conclude that, for an appropriate value of j_0 and for $|\alpha-\beta|\geqslant 2^{-j_0}$, $l_{\alpha\beta}$ can be neglected, and that we may make analogous approximations for $n_{\alpha\beta\gamma}$. Using the fact that KS is translationally invariant, we can replace each of the moments in Eq. (12) by the analogous quantity for which all indices belong to the same B_k box. For instance,

$$\langle a_\alpha(t)a_\beta(t+\tau)\rangle = \langle a_{\alpha-\Delta}(t)a_{\beta-\Delta}(t+\tau)\rangle, \tag{13}$$

where $(\alpha-\Delta),(\beta-\Delta)\in B_k$ and Δ is the translation parameter. Finally, using the approximate relation (8), it is easy to obtain the scaling relations between the coupling coefficients $l_{\alpha\beta}$ and $n_{\alpha\beta\gamma}$ for Eq. (1) with parameter L and $\tilde{L}=L2^{-(j_0+1)/2}$

$$l_{\alpha\beta}(L)=l_{\alpha_1\beta_1}(\tilde{L})[1+O(\exp^{-2^{j_{\min}/S}})],$$

$$n_{\alpha\beta\gamma}(L)=2^{-(j_0+1)/2}n_{\alpha_1\beta_1\gamma_1}(\tilde{L})[1+O(\exp^{-2^{j_{\min}/S}})], \tag{14}$$

where $\alpha=(j,k),\alpha_1=(j_1,k_1)$, $j_1=j-(j_0+1)$, $k_1=2^{-(j_0+1)}k$, etc., $\ldots$, and $j_{\min}=\min(j,j',j'')$. Using these relations, one can easily see that the rescaled variables $\tilde{a}_{\alpha_1}=2^{(j_0+1)/2}a_\alpha$ approximately satisfy the moment equations (12) arising from Eq. (1) for length parameter $\tilde{L}$. Assembling all these observations, one gets the relation claimed above, between the equation governing the statistical properties of solutions of (1) with parameter $\tilde{L}$ and those of (11) with no forcing. However, close equations do not in general imply close solutions, even in a statistical sense. Consequently, we do not have any *a priori* reason to believe that the statistical properties of the small scales of Eq. (1) with parameter L will be close to those with parameter $\tilde{L}$, even if the forcing terms $\delta l^{>>}$ and $f^>$ are small. In the next section we investigate this question using numerical simulations.

III. COUPLING SMALL AND LARGE SCALES

In this section, we investigate by means of numerical simulations how the approximations performed in the preceding section on the unforced small-scale equations (11), survive when one includes the slow forcing terms coming from the interaction with the large scales. Our intuition is that these approximations can hold provided the amplitude of the forcing is small and its evolution well separated in time from the small scales. The same idea, rephrased in the context of the ''integrated'' KS equation for the function $h(x,t)=\int_0^x u(x',t)dx'$, constitutes the main ansatz of the work by Chow and Hwa [9]. As we noted above, the second condition is automatically satisfied by wavelet decompositions of the KSE in the region of the spectrum we are interested in. On the other hand, the amplitude of the forcing is essentially determined by the value of the cutoff j_0. As already observed in previous works, the energy of a typical

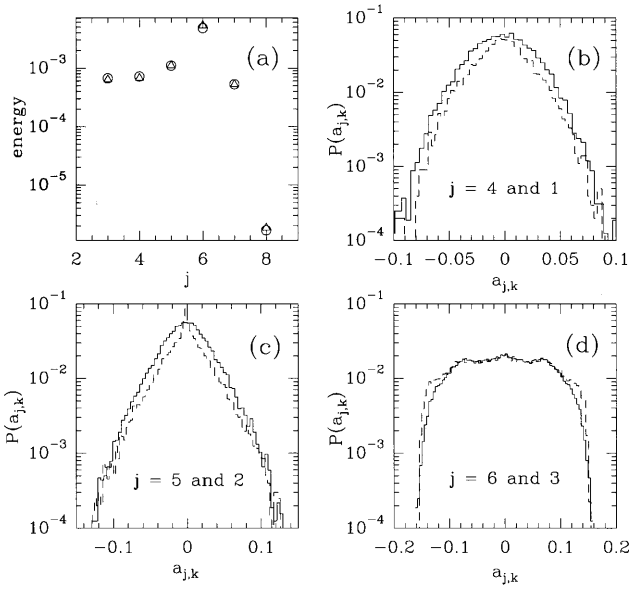

FIG. 1. (a) Time average of a_α^2 as a function of j: ($\triangle$), $L=400$, ($\bigcirc$) $L=50$, divided by $8=2^{j_0+1}$. (b), (c), and (d) present respectively, the pdf's $P(a_{j,k=0}$ of the wavelet coefficients $a_{j,k=0}$ for scales: $j=4$, 5, and 6 computed from the KS equation with $L=400$ (solid line), and $j=1$, 2, and 3 for $L=50$ (dashed line), rescaled by a factor $2\sqrt{2}$.

solution of the KS equation is mainly concentrated around the wavelength $2\pi\sqrt{2}/L$, corresponding to the most unstable Fourier mode of the equation linearized about the trivial solution. In wavelet terms, this corresponds to the value of j_M such that $2^{-(j_M+1)}<2\pi\sqrt{2}/L<2^{-j_M}$. Thus we expect that the above scaling relations between statistical quantities obtained for two different values of L will hold even in the presence of the slow forcing terms $\delta l^{>>}$ and $f^{>}$, provided that $j_0<j_M$. In order to illustrate this point, we present in Fig. 1 the second moments and the probability distribution functions of the wavelet coefficients $a_{j,k}$ for several scales j and for the two parameter values $L=50$ and $L=400$ (that is, $j_M\approx 5$ and $j_0=2$), with the appropriate rescaling. The most energetic scale $j=6$ as well as the second moments of these distributions are in a rather good agreement. This means that (i) the scaling relations obtained in Sec. II for the unforced equations do imply approximate scaling relations for the statistical properties of the solutions for length values L and $\tilde{L}$, and (ii) the forcing is weak enough to preserve the relations obtained for the unforced equations. Notice however that the same comparison with $j_0=3$ would yield a very poor agreement. The reason is that the asymptotic dynamics for the KS equation with $L=25$ is a fixed point (cf. [1]): this system is simply too ''short'' to exhibit sufficiently rich dynamics *in isolation* (but see below as well as Sec. IV). We conclude that the choice of j_0 is actually dictated by at least two conditions: (i) the order of magnitude of the forcing terms $\delta^{>>}$ and $f^{>}$, and (ii) the (asymptotic) dynamics of the KS equation for the parameter value $\tilde{L}$.

Let us now focus on the large-scale equations of motion

$$\dot{a}_\alpha^{<}=[l_{\alpha\beta}^{<<}+\delta l_{\alpha\beta}^{<<}(t)]a_\beta^{<}+n_{\alpha\beta\gamma}^{<<<}a_\beta^{<}a_\gamma^{<}+f_\alpha^{<}(t), \quad (15)$$

where $\delta l_{\alpha\beta}^{<<}(t)=n_{\alpha\beta\gamma}^{<<>}a_\gamma^{>}+n_{\alpha\gamma\beta}^{<><}a_\gamma^{>}$ and $f_\alpha^{<}(t)=n_{\alpha\beta\gamma}^{<>>}a_\beta^{>}a_\gamma^{>}+l_{\alpha\beta}^{<>}a_\beta^{>}$. In the standard Fourier-based RNG procedure [6,16], the term $f_\alpha^{<}$ gives rise to the correction of the linear term (renormalization of the viscosity), whereas $\delta l_{\alpha\beta}^{<<}$ is either neglected, or shown [16] to correspond to a cubic nonlinearity only significant for values of j close to j_0. The common procedure used to reach these results is, in one form or another, the slaving principle. For instance, one performs a ''short time'' average of (15) in order to replace $a_\alpha^{>}$ by $\langle a_\alpha^{>}\rangle$ ($\{a^{<}\}$) and $a_\alpha^{>}a_\beta^{>}$ by some function $\langle a_\alpha^{>}a_\beta^{>}\rangle$ ($\{a^{<}\}$). Different versions of the RNG procedure in fact give different analytic expressions for the averaged terms $\langle a_\alpha^{>}\rangle$ ($\{a^{<}\}$) and $\langle a_\alpha^{>}a_\beta^{>}\rangle$ ($\{a^{<}\}$).

Following our initial picture of the small scales as a set of 2^{j_0+1} independent boxes slowly driven by the large-scale variables, we wish to check its validity with regard to the statistics of the forcing terms $\delta l_{\alpha\beta}^{<<}$ and $f_\alpha^{<}$. With this in mind, we performed three kinds of simulations. Simulation 1 is just the integration of the full KS equation for $L=400$, using a pseudospectral code, with Adams-Bashforth time stepping for the nonlinear term, and a Crank-Nicholson scheme for the linear term. Simulation 2 is integration in time of $2^{j_0+1}(=8)$ *independent* KS equations with $L=50$, each forced by the set of large-scale ($j\leqslant j_0$) components $a_\alpha^{<}$ computed from simulation 1. Namely, for $k=0,\ldots,2^{j_0+1}-1$, we integrate (using a fourth order Runge-Kutta method) the equations

$$\begin{aligned}\dot{a}_\alpha^{>}=&\sum_{a_\beta^{>}\in B_k} l_{\alpha\beta}^{>>}(\tilde{L})a_\beta^{>}+2^{-(j_0+1)/2}\sum_{a_\beta^{>},a_\gamma^{>}\in B_k} n_{\alpha\beta\gamma}^{>>>}(\tilde{L})a_\beta^{>}a_\gamma^{>}\\&+\sum_{a_\beta^{>}\in B_k,a_\gamma^{<}}[n_{\alpha\beta\gamma}^{>><}(L)+n_{\alpha\gamma\beta}^{><>}(L)]a_\gamma^{<}\\&+\sum_{a_\beta^{<}a_\gamma^{<}} n_{\alpha\beta\gamma}^{>><}(L)a_\beta^{<}a_\gamma^{<}. \quad (16)\end{aligned}$$

[Here we momentarily drop the implicit summation convention, and make use of the approximate scaling relation (14).] The comparison between simulations 1 and 2 will substantiate the main point of this paper, namely, that the statistics of couplings between small and large scales in Eq. (1) can be computed from the interaction with an assembly of low dimensional systems. Simulation 3 will be described at the end of this section.

We considered the statistics of the two $[a_\alpha(t)a_\beta(t-\tau)]$ and three $[a_\alpha(t)a_\beta(t)a_\gamma(t-\tau)]$ point products. These enter for instance in the energy budget equation ($\tau=0$)

$$\begin{aligned}\frac{da_\alpha^{<2}}{dt}&=l_{\alpha\beta}^{<<}a_\alpha^{<}a_\beta^{<}+n_{\alpha\beta\gamma}^{<<<}a_\alpha^{<}a_\beta^{<}a_\gamma^{<}+\delta l_{\alpha\beta}^{<<}a_\alpha^{<}a_\beta^{<}+f_\alpha^{<}a_\alpha^{<}\\&\equiv l_{\alpha\beta}^{<<}a_\alpha^{<}a_\beta^{<}+T_\alpha^{<<}+T_\alpha^{<>}+T_\alpha^{>>}, \quad (17)\end{aligned}$$

and have been studied in the RNG approach to the Navier-Stokes equations [16,17]. The time average of the transfer $T_\alpha^{>>}$ is shown [16,17] to correspond to a negative linear correction to the $l_{\alpha\alpha}\langle a_\alpha^2\rangle$ term. As can be seen in Fig. 2, the average $\langle T_\alpha^{>>}\rangle$ is negative, although very frequently the ''in-

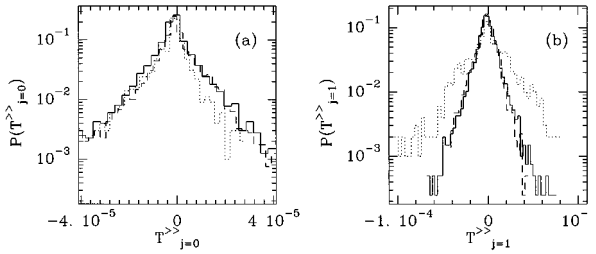

FIG. 2. (a) Probability distribution function of the transfer $T_\alpha^{>>}$ term for $j=0$, corresponding to the integration of the full KS equation (solid line), simulation 2 (long-dashed line), and simulation 3 (short-dashed line), for $j_0=2$. (b) Same as in (a) for the $j=1$ transfer $T_\alpha^{>>}$.

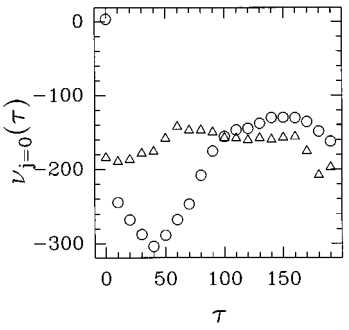

FIG. 3. $\nu_\alpha(\tau)$ computed from the data obtained with both simulations 1 (○) and 2 (△).

stantaneous'' transfer is actually positive. Simulations 1 and 2 seem to agree fairly well for any $j \leqslant j_0$. The same remark is also valid for the other values of α, as well as for the ''cross'' transfer term $T_\alpha^{<>}$. The good agreement between the two statistics stresses the fact that all the information needed to compute the interaction between small and large scales (up to a good approximation) is actually encoded in the dynamics of the individual B_k boxes.

Delayed two-point correlations ($\tau>0$) are also of interest. They enter, for instance, in the analogue of the Zaleski's procedure [3,8] written in wavelet basis. Namely, the correction to the viscosity $\delta\nu_\alpha$ and the stochastic forcing $F_\alpha^>$ are defined by the following equation:

$$\begin{aligned}\dot{a}_\alpha^< &= [l_{\alpha\beta}^{<<} + \delta l_{\alpha\beta}^{<<}(t)]a_\beta^< + n_{\alpha\beta\gamma}^{<<<} a_\beta^< a_\gamma^< + f_\alpha^<(t) \\ &= (l_{\alpha\beta}^{<<} + \delta\nu_\alpha\delta_{\alpha\beta})a_\beta^< + n_{\alpha\beta\gamma}^{<<<} a_\beta^< a_\gamma^< + F_\alpha^<, \end{aligned} \tag{18}$$

where

$$F_\alpha^< = -\delta\nu_\alpha a_\alpha^< + \delta l_{\alpha\beta}^{<<}(t)a_\beta^< + f_\alpha^< \tag{19}$$

(note that no summation is assumed in the $\delta\nu_\alpha a_\alpha^<$ term). The closure equation is given by the condition that $F_\alpha^<(t)$ and $u_\alpha^<(t-\tau)$ are uncorrelated for $\tau>0$. This yields the following condition:

$$\begin{aligned}\delta\nu_\alpha &= \lim_{\tau\to\infty} \delta\nu_\alpha(\tau) \\ &= \lim_{\tau\to\infty} \frac{\langle a_\alpha(t-\tau)[f_\alpha^<(t) + \delta l_{\alpha\beta}^{<<}(t)a_\beta(t)]\rangle}{\langle a_\alpha(t-\tau)a_\alpha(t)\rangle}. \end{aligned} \tag{20}$$

Notice that $\delta\nu_\alpha$ is a correction of the diagonal matrix elements of $l_{\alpha\beta}$, and that a more general model for the correction to the viscosity could be given by

$$\dot{a}_\alpha^< = (l_{\alpha\beta}^{<<} + \delta\nu_{\alpha\beta})a_\beta^< + n_{\alpha\beta\gamma}^{<<<} a_\beta^< a_\gamma^< + \tilde{F}_\alpha^<. \tag{21}$$

Due to the lack of precision in the numerical estimation of $\delta\nu_\alpha$ (see Fig. 3), it is difficult to decide which of the above models is the best (in the sense that the limit as $\tau\to\infty$ is reached faster), and we will stick to the simplest, namely, Eq. (18). Figure 3 represents $\delta\nu(\tau)_\alpha$ computed from the data obtained from both simulations 1 and 2. One can see that, in the asymptotic range $\tau>100$, our model reproduces reasonably well data obtained from the full simulation, although the rate of convergence does not seem to be the same for both simulations.

In Ref. [9], Chow and Hwa used a rather sophisticated method to measure the response function of the drift rate, which, in our terminology, basically gives the correction to the viscosity. They apply at $t=0$ a constant (in time) forcing to the solution of the integrated KS equation in a periodic small box. The spatial form of the forcing is critical in order to obtain the correct value of the response function, and is thoroughly discussed in their paper. One should notice that our way of obtaining the correction to the viscosity is analogous to theirs, excepting the fact that the external excitation of the small periodic boxes is actually given by the real forcing computed from the full simulation. This is probably not important for the scales much larger than the cutoff scale j_0, but probably critical for values of $j\sim j_0$. Notice also that our procedure yields unambiguously the spatial profile of the forcing, simply given by the $\delta l_{\alpha\beta}^{>>}$ and $f_\alpha^>$ terms.

It is interesting to note that similar results can be obtained for the parameter value $j_0=3$, which corresponds to $L=25$ for each box B_k. In terms of dynamical systems, in this case the asymptotic behavior of the KS equation for $L=25$ is much simpler than for $L=50$ (the global attractor is a trimodal steady state: see [1] for details). Thus, one cannot compare the pdf's of wavelet coefficients as we did in Fig. 1, although it should be noted that for such a value of L, the *transient* dynamics of the KS equation can be rather complicated, and pdf's computed on a time interval where only the transients are present are actually not very different from those in Fig. 1. Then, the forcing from the large scales permanently keeps the small scales away from the trimodal fixed point in such a manner that the energy transfer statistics are those of simulation 1. Hence we conclude that the *statistical* properties of the *forced* KS equation for $L=25$ are sufficient to describe those of the same equation for larger values of L. However, if one tries to increase the value of j_0 to 4, that is, to reduce to $L=12.5$ for the B_k boxes, the dynamics of the resulting short system are insufficiently rich to reproduce those for larger values of L (the solutions quickly converge to traveling waves with not enough long transients). Although this choice still satisfies the condition $j_0<j_M\sim 5$, the results of simulation 2 are very different from those of 1.

The conclusion of the two simulations presented so far is clear: forcing the set of 2^{j_0+1} replicas of the KS equation

with the right statistics of the large scales induces the correct energy transfer between small and large scales. We now wish to investigate the effective equation suggested by our approach. To do this, we couple the small-scales equations (16) with the following set of equations for the large scales:

$$\dot{a}^{<}_{\alpha} = \sum_{a^{<}_{\alpha}} l^{<<}_{\alpha\beta}(L) a^{<}_{\alpha} + \sum_{a^{<}_{\beta} a^{>}_{\gamma}} n^{<<<}_{\alpha\beta\gamma}(L) a^{<}_{\beta} a^{<}_{\gamma} + \sum_{a^{<}_{\beta} a^{>}_{\gamma}} [n^{<<>}_{\alpha\beta\gamma}(L) + n^{<><}_{\alpha\gamma\beta}(L)] a^{<}_{\beta} a^{>}_{\gamma} + \sum_{a^{>}_{\beta} a^{>}_{\gamma}} n^{<>>}_{\alpha\beta\gamma}(L) a^{>}_{\beta} a^{>}_{\gamma} + \sum_{a^{>}_{\beta}} l^{<>}_{\alpha\beta}(L) a^{>}_{\beta} . \quad (22)$$

Numerical simulations using the coupled set of Eqs. (16) and (22) will be called simulation 3 hereafter. We used the same time stepping as in simulation 2, and also tried a slightly different version of simulation 3, where we computed the evolution of the ''top'' coefficient $a_{j=j_0+1,k}$ of each box B_k using Eq. (22) instead of Eq. (16). This version seems to yield better results than the former, and will actually be used in the sequel. This improvement can be easily understood, as the error in the approximate scaling relations (14) decreases as j increases from j_0+1.

The ensemble formed by Eqs. (16) for $k=0,1,\ldots,2^{j_0+1}-1$ and (22) constitutes a completely autonomous model for the statistics of the KS equation (neither fitting parameters other than the cutoff scale j_0 nor external forcing terms are needed). The comparison (Fig. 2) of the undelayed three-point correlation functions with those of simulations 1 and 2 shows clearly that, as one could expect, the agreement deteriorates as j approaches j_0. The pdf's of the transfer term are close for $j=0$, deviate for strong (but rare) values of the transfer for $j=1$, and differ by a factor ~ 4 for $j=2$ (not shown in Fig. 2). Such a disagreement should be expected: as a matter of fact, the periodization approximation involved in our model neglects nonlinear couplings of the type $n^{<>>}_{\alpha\beta\gamma}$, where the modes Ψ_β and Ψ_γ belong to different boxes, and α corresponds to some large-scale mode. Simulation 2 shows that this approximation is correct as soon as the small scales are forced with the right statistics. On the other hand, simulation 3 shows that these missing interactions can significantly change the statistics of the large scales close to the cutoff scale j_0.

IV. CONCLUSION

In this paper, we have used orthogonal wavelet projections to study models for the transfer of energy between small and large scales in the 1D periodic Kuramoto-Sivashinsky equation. We first showed that the structure of the small-scale equations is such that, when neglecting couplings with the large scales, they can be approximated in a natural way by a set of 2^{j_0+1} replicas of the wavelet projection of Eq. (1) for length parameter $\tilde{L}=2^{-(j_0+1)}L$. This approximation is intended only to preserve the statistics of the solutions, and not the ''instantaneous'' dynamics.

By means of numerical simulations, we then checked that forcing this set of 2^{j_0+1} independent subsystems with the large scales computed from the full simulation of Eq. (1) does actually reasonably reproduce the energy transfer between large and small scales, provided the phase space of each of the uncoupled boxes B_k is large enough. However, when the large scales are generated autonomously by a ''closed'' model coupled to the independent subsystems, the agreement is satisfactory only for scales separated from the cutoff scale j_0.

The statistical picture suggested by our study is that of an assembly of identical short length subsystems, slowly driven via interactions with the large scales. A more detailed study [18] of the dynamics of the KS equation for values of $L\sim 400$ actually suggests the need for at least two kinds of solutions in these subsystems, corresponding, respectively, to traveling waves and homoclinic cycles, in order to obtain reasonable tracking of the solutions of the full equation. Spatiotemporal representations of the solutions of the $L=50$ KS equation strongly suggest that the latter does indeed exhibit these two types of dynamics.

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